# Derivation of the weak form

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$\overset{\Rightarrow}{\sigma}$: Stress tensor
$\vec{w}$: Vector of test functions
$\vec{n}$: Normal vector
$\lambda, \mu$: Lamé-constants
$\vec{u}$: Displacement ector
$\overset{\Rightarrow}{I}$: Identity tensor
$\vec{t}$: Surface-traction vector

$\mathrm{div}(.) \hat{=} \nabla\cdot(.)$
$\mathrm{grad}(.) \hat{=} \nabla()$

Equilibrium Conditions:

$$\mathrm{div}(\overset{\Rightarrow}{\sigma}) = \vec{0}$$

Multiplying with a vector of test functions and integrating over the domain:

$$\int_V \vec{w}\cdot \mathrm{div}(\overset{\Rightarrow}{\sigma})\,dV = 0$$

Using Green's formular / integration by parts

$$\int_V \mathrm{grad}(\vec{w}) \cdot\cdot \overset{\Rightarrow}{\sigma}\,dV - \int_A \vec{w}\cdot(\overset{\Rightarrow}{\sigma}\cdot\vec{n})\,dA = 0$$

Using Hooke's Law to express $\sigma$ in terms of displacements:

$$\int_V \nabla w \cdot \left[\lambda(\nabla\cdot u)\mathbb{1} + \mu\left(\nabla u + (\nabla u)^T\right)\right]dV = \int_A w\cdot t\,dA$$

$$\int_V \mathrm{grad}(\vec{w})\cdot\left[\lambda\,\mathrm{div}(\vec{u})\overset{\Rightarrow}{I} + \mu\left[\mathrm{grad}(\vec{u}) + (\mathrm{grad}(\vec{u}))^T\right]\right]dV = \int_A \vec{w}\cdot\vec{t}\,dA$$

In Comsol I am inserting the Left-hand-side term as a weak form - this will be of interest in the next section.

Evaluating the differential operators in cylindrical / axial symmetric coordinates:

$$\mathrm{grad}(\vec{w}) \underbrace{=}_{\text{cylindrical}} \begin{pmatrix} \frac{\partial w_r}{\partial r} & \frac{1}{r}\left(\frac{\partial w_r}{\partial \varphi} - w_\varphi\right) & \frac{\partial w_r}{\partial z} \\ \frac{\partial w_\varphi}{\partial r} & \frac{1}{r}\left(\frac{\partial w_\varphi}{\partial \varphi} + w_r\right) & \frac{\partial w_\varphi}{\partial z} \\ \frac{\partial w_z}{\partial r} & \frac{1}{r}\frac{\partial w_z}{\partial \varphi} & \frac{\partial w_z}{\partial z} \end{pmatrix}_{\vec{e}_r, \vec{e}_\varphi, \vec{e}_z} \underbrace{=}_{\text{axial symmetry}} \begin{pmatrix} \frac{\partial w_r}{\partial r} & 0 & \frac{\partial w_r}{\partial z} \\ 0 & \frac{w_r}{r} & 0 \\ \frac{\partial w_z}{\partial r} & 0 & \frac{\partial w_z}{\partial z} \end{pmatrix}_{\vec{e}_r, \vec{e}_\varphi, \vec{e}_z}$$

$$\mathrm{div}(\vec{u}) \underbrace{=}_{\text{cylindrical}} \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z} \underbrace{=}_{\text{axial symmetry}} \frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}$$

Inserting these relations into the equations gives:

$$\int_V \left[\lambda\left(\frac{w_r}{r} + \frac{\partial w_r}{\partial r} + \frac{\partial w_z}{\partial z}\right)\left(\frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}\right)\right.$$

$$+\mu \begin{pmatrix} \frac{\partial w_r}{\partial r} & 0 & \frac{\partial w_r}{\partial z} \\ 0 & \frac{w_r}{r} & 0 \\ \frac{\partial w_z}{\partial r} & 0 & \frac{\partial w_z}{\partial z} \end{pmatrix} : \begin{pmatrix} 2\frac{\partial u_r}{\partial r} & 0 & \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\ 0 & 2\frac{u_r}{r} & 0 \\ \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) & 0 & 2\frac{\partial u_z}{\partial z} \end{pmatrix} \Bigg] dV$$

$$= \int_V \Bigg[ \frac{w_r}{r}\left(\lambda \frac{u_r}{r} + \lambda \frac{\partial u_r}{\partial r} + \lambda \frac{\partial u_z}{\partial z} + 2\mu \frac{u_r}{r}\right) + \frac{\partial w_r}{\partial r}\left(\lambda \frac{u_r}{r} + \lambda \frac{\partial u_r}{\partial r} + \lambda \frac{\partial u_z}{\partial z} + 2\mu \frac{\partial u_r}{\partial r}\right) + \mu \frac{\partial w_r}{\partial z}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

$$+ \mu \frac{\partial w_z}{\partial r}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) + \frac{\partial w_z}{\partial z}\left(\lambda \frac{u_r}{r} + \lambda \frac{\partial u_r}{\partial r} + \lambda \frac{\partial u_z}{\partial z} + 2\mu \frac{\partial u_z}{\partial z}\right)\Bigg] dV$$

In Comsol-Notation this gives ($u_r = u1$, $u_z = u2$):

$$\int_V \Bigg[ \frac{test(u1)}{r}\left((\lambda+2\mu)\frac{u1}{r} + \lambda(u1r + u2z)\right) + test(u1r)\left((\lambda+2\mu)u1r + \lambda\left(\frac{u1}{r} + u2z\right)\right) + \mu\, test(u1z)(u1z + u2r)$$

$$+ \mu\, test(u2r)(u1z + u2r) + test(u2z)\left((\lambda+2\mu)u2z + \lambda\left(\frac{u1}{r} + u1r\right)\right)\Bigg] dV$$

The first line is enterred in the upper window for the weak form;
the second line is enterred in the lower window for the weak form.