Project 1

Due April 15, 2016

(PDF files only, 4 pages max)

PROBLEM 1

We are interested in the analysis of a column of length L, cross-sectional area A, and Young's modulus E. We assume that the column stands on a support at $x = 0$, that it is subjected to a longitudinal compression force P at $x = L$ and to the gravitational force density g. The displacement $u = u(x)$ in the column is governed by the 1D differential equation:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right) = -\rho g A, \quad \text{in } (0, L)$$

and subjected to the Dirichlet and Neuman BCs:

$$u = 0, \quad \text{at } x = 0, \qquad \text{and} \quad EA\frac{\mathrm{d}u}{\mathrm{d}x} = -P, \quad \text{at } x = L$$

The following data will be the same for all questions: $L = 4$ m, $g = 9.81$ m/s^2, $P = 40$ kN.

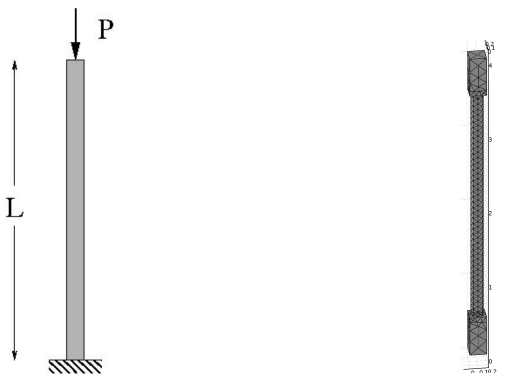

Figure 1: Description of the problem in 1D (left) and example of mesh for the 3D model (right, Problem 2.

1.1) In this question, take E, A, and ρ constant along x: $E = 20$ GPa, $\rho = 2300$ kg/m^3 (corresponding to concrete), and $A = A_0 = 0.0341$ m^2.

1. Solve for the exact solution and derive the weak formulation of the problem.
2. Develop an application in Comsol Multiphysics to model the problem.
3. Compute the stress $\sigma = E\mathrm{d}u/\mathrm{d}x$ and the relative error in the stress at $x = 0$ when using 1, 2, 4, 8, and 16 linear elements of uniform size.
4. Using non uniform linear elements, design, by trial and error, a mesh that yields the minimal number of degrees of freedom while reaching a relative error in the stress at $x = 0$ smaller than half a percent.

1.2) Keep here E and ρ constant ($E = 20$ GPa, $\rho = 2300$ kg/m^3), and consider A such that:

$$A = A_0 \left[1 - \frac{x(L-x)}{L^2} \right]$$

with $A_0 = 0.0341$ m^2. Here, we do not want to spend time on deriving the exact solution: instead, we prefer to compute what we call an "overkill" solution, that is a numerical solution computed on a very fine mesh[1] (*e.g.* several hundreds of elements here).
Repeat questions 2, 3, 4 of Part 1.1.

1.3) Suppose that the column is made of two different materials:

- in regions $(0, l)$ and $(L - l, L)$, with $l = 50$ cm, the column is made of a material with properties $E = 10$ GPa and $\rho = 500$ kg/m^3, and in these two regions, the column has a constant square cross-section with width $a_0 = 0.20$ m;
- in region $(l, L - l)$, the column has material properties $E = 20$ GPa, $\rho = 2300$ kg/m^3, and a constant circular cross-section with diameter $d_0 = 0.15$ m.

Find the location x_s where the stress is maximal in the column. Design a mesh that should give a relative error in the maximal stress smaller than one percent.

PROBLEM 2

Develop a 3D FE model using linear elasticity to simulate Part 1.3 of Problem 1 (one will use here a Poisson's ratio $\nu = 0.3$ for both materials). Suppose that the different components of the column are perfectly aligned along the centerline and that the force P is equally distributed at $x = L$.
Find the maximal stress σ_s and corresponding location x_s in the column (make sure that the mesh is sufficiently refined to provide an accurate solution). Compare with the 1D solution computed above.

[1] We will see later that, under some assumptions, the Finite Element Method converges towards the exact solution as the number of degrees of freedom tends to infinity.

PROBLEM 3

Suppose now that the circular column was imperfectly aligned with respect to the two other blocks by $\delta = 0.02$ m. Using 3D linear elasticity and assuming that the force P is equally distributed at $x = L$, compute the maximal deflection of the column.