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Application of Kelvin's Inversion Theorem to the Solution of Laplace's Equation over a Domain that Includes the Unbounded Exterior of a Sphere

John M. Russell, Professor Emeritus
Florida Inst. of Technology, Melbourne, Florida
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Objectives

1. **Formulate** a numerical method that fulfills two requirements, namely
   i. The method solves for a harmonic function over an unbounded domain; and
   ii. The only computational domains for which the method requires meshing are bounded; and
2. **Test** the computational results for agreement with an analytic solution.

Kelvin inversion

Let a be a given length scale, let $\mathbf{r}$ denote position relative to an origin, and let $r = |\mathbf{r}|$. Consider the change of position variable $\mathbf{r} \to \mathbf{q}$ defined by the rule $\mathbf{r}/r = \mathbf{q}/q$ with $rq = a^2$ and $q = |\mathbf{q}|$. Then a is the geometric mean of r and q. If $q < r$ then $q < a < r$.

If, in particular, $\mathbf{r}$ is a point exterior to the sphere $r = a$ then $\mathbf{q}$ is a point interior to that sphere. Points $\mathbf{r}$ and $\mathbf{q}$ are *Kelvin inverses* of one another. I will call the sphere $r = a = q$ the *Reflecting Sphere.*

Kelvin's Inversion Theorem

Let (x_1, x_2, x_3) and (q_1, q_2, q_3) denote the cartesian components of $\mathbf{r}$ and $\mathbf{q}$, resp., relative to a common set of cartesian unit vectors $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ and let $\Delta_x := \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j \partial x_j}$ and $\Delta_q := \sum_{j=1}^{3} \frac{\partial^2}{\partial q_j \partial q_j}$. One may show that for any suitably differentiable function $(x_1, x_2, x_3) \mapsto \phi$ and $q \neq 0$ we have $\Delta_x \phi = (q/a)^5 \Delta_q[(a/q)\phi]$. If, therefore, $q \neq 0$ and $\Delta_x \phi = 0$ then $\Delta_q \Phi = 0$, in which $\Phi := (a/q)\phi$.

A boundary-value problem

Let a_s be the radius of a solid sphere. If the sphere is submerged in a liquid initially at rest and accelerated suddenly to the velocity $\mathbf{Q}$ then the velocity field, $\mathbf{r} \mapsto \mathbf{v}$ of the fluid satisfies the equations $\nabla_x \bullet \mathbf{v} = 0$ (incompressibility of the fluid) and $\mathbf{v} = \nabla_x \phi$ (irrotationality of the motion). Then $\nabla_x \bullet \nabla_x \phi = \Delta_x \phi = 0$. Suitable boundary conditions are: (i) $|\mathbf{v}| = |\nabla_x \phi| \longrightarrow 0$ as $r \longrightarrow \infty$; and (ii), $(\nabla_x \phi \bullet \hat{\mathbf{n}})_{r=a_s} = \mathbf{Q} \bullet \hat{\mathbf{n}}$, in which $\hat{\mathbf{n}}$ is the outward unit normal for the exterior of the solid sphere.

Analytic solution of the boundary-value problem

If (Q_1, Q_2, Q_3) are the components of $\mathbf{Q}$ relative to $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ one may orient the latter such that $\mathbf{Q} \bullet \hat{\mathbf{i}}_3 = Q_3$. According to a classical result the function $\mathbf{r} \mapsto \phi_d$ defined by

$$\phi_d = -(1/2)(a_s/r)^3 Q_3 x_3 \ ,$$

in which $(\ )_d$ stands for for *dipole*, is an exact solution of the foregoing boundary-value problem. This solution will serve as the benchmark against which to compare the numerical simulation.

Decomposition of the domain

Given a_s choose a such that $a/a_s > 1$. I will call the regions with $a_s < r < a$ and $r > a$ the *Near Exterior* and the *Far Exterior*, resp. Then Kelvin inversion $\mathbf{r} \mapsto \mathbf{q}$ takes points with position $\mathbf{r}$ in the Far Exterior to points with position $\mathbf{q}$ in the ball $q < a$, which I will call the *Inverted Far Exterior*.

The idea is to solve $\Delta_x \phi = 0$ and $\Delta_q \Phi = 0$ simultaneously for the functions $\mathbf{r} \mapsto \phi$ in the Near Exterior and $\mathbf{q} \mapsto \Phi$ in the Inverted Far Exterior, resp.

Impermeable-wall condition and the first compatibility condition

As stated earlier the impermeable-wall boundary condition is $(\nabla_x \phi \bullet \hat{\mathbf{n}})_{r=a_s} = \mathbf{Q} \bullet \hat{\mathbf{n}}$ on the inner boundary of the Near Exterior.

Recall that $q = a = r$ on the Reflecting Sphere and recall the definition $\Phi := (a/q)\phi$. Therefore $\phi = (q/a)\Phi$, which leads to the *first compatibility condition* at the Reflecting Sphere, namely

$$\phi_{r=a} = \Phi_{q=a} \ .$$

The second compatibility condition

If one differentiates $\phi = (q/a)\Phi$ with respect to q one obtains $\partial\phi/\partial q = (q/a)(\partial\Phi/\partial q) + (1/a)\Phi$. But $\partial\phi/\partial q = (dr/dq)\partial\phi/\partial r$ and $rq = a^2$, from which one deduces that

$$-(a^2/q^2)(\partial\phi/\partial r) = (q/a)(\partial\Phi/\partial q) + (1/a)\Phi \; .$$

If one evaluates on the Reflecting Sphere and rearranges one obtains

$$(\partial\Phi/\partial q)_{q=a} = -(\partial\phi/\partial r)_{r=a} - (1/a)\Phi_{q=a} \; ,$$

which is the *second compatibility condition* at the Reflecting Sphere.

Remarks on on the COMSOL implementation, I

Consider cylinderical coordinates (ϖ, φ, z) associated with the cartesian coordinates (x_1, x_2, x_3) through $x_1 = \varpi \cos\varphi$, $x_2 = \varpi \sin\varphi$, $x_3 = z$. A classical result asserts that in the axisymmetric case $\Delta_x \phi = (1/\varpi)(\varpi\phi_\varpi)_\varpi + \phi_{zz}$, or, equivalently, $\varpi\Delta_x\phi = (\varpi\phi_\varpi)_\varpi + (\varpi\phi_z)_z$. The aim is to solve $\Delta_x\phi = 0$, so

$$(\varpi\phi_\varpi)_\varpi + (\varpi\phi_z)_z = 0\ ,$$

which COMSOL recognizes as a classical PDE of Poisson (not Laplace) type.

Remarks on on the COMSOL implementation, II

To enable each of the two model branches to read data from the other I introduced, in the local Definitions list of each model, a Model Coupling Operator of General Extrusion type (at the boundary level) for use on the Reflecting Sphere.

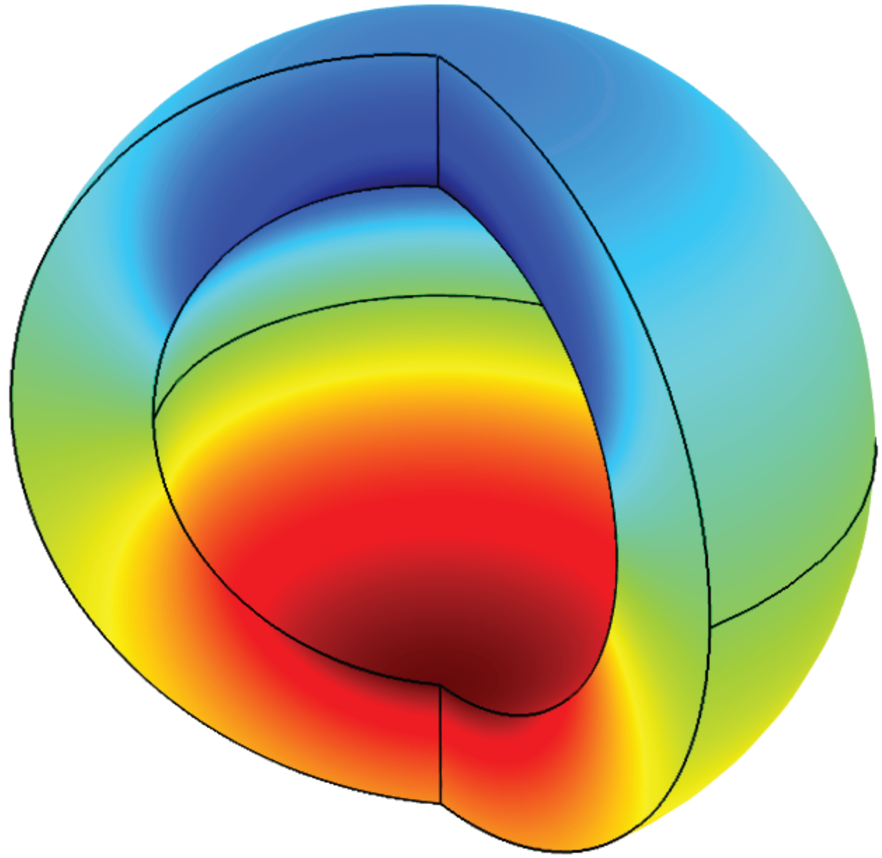

Fig. 1 Computed velocity potential, ϕ, in the Near Exterior with $a/a_s = 1.5$ and default mesh in both models. The range for ϕ is $(-0.5, 0.5)\,\mathrm{m}^2/\mathrm{s}$.

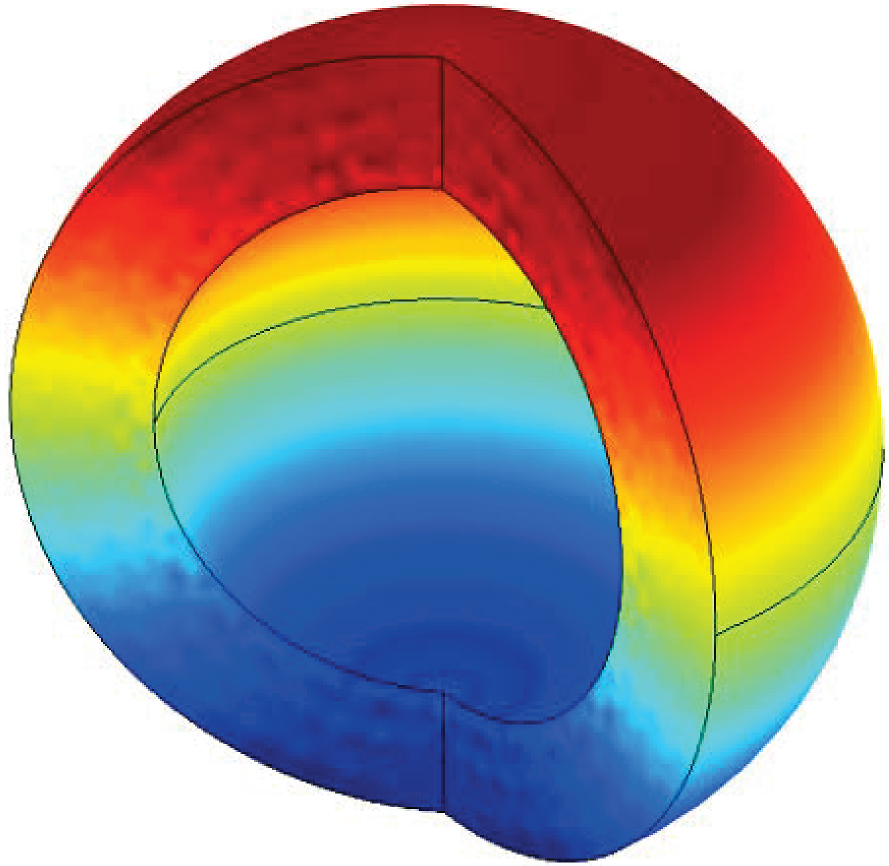

Fig. 2 Discrepancy, $\Delta\phi := \phi - \phi_{\mathrm{analytic}}$, between COMSOL and analytical calculation in the Near Exterior.The range of $\Delta\phi$ is $(-2.63, 2.99) \times 10^{-3}\,\mathrm{m}^2/\mathrm{s}$.

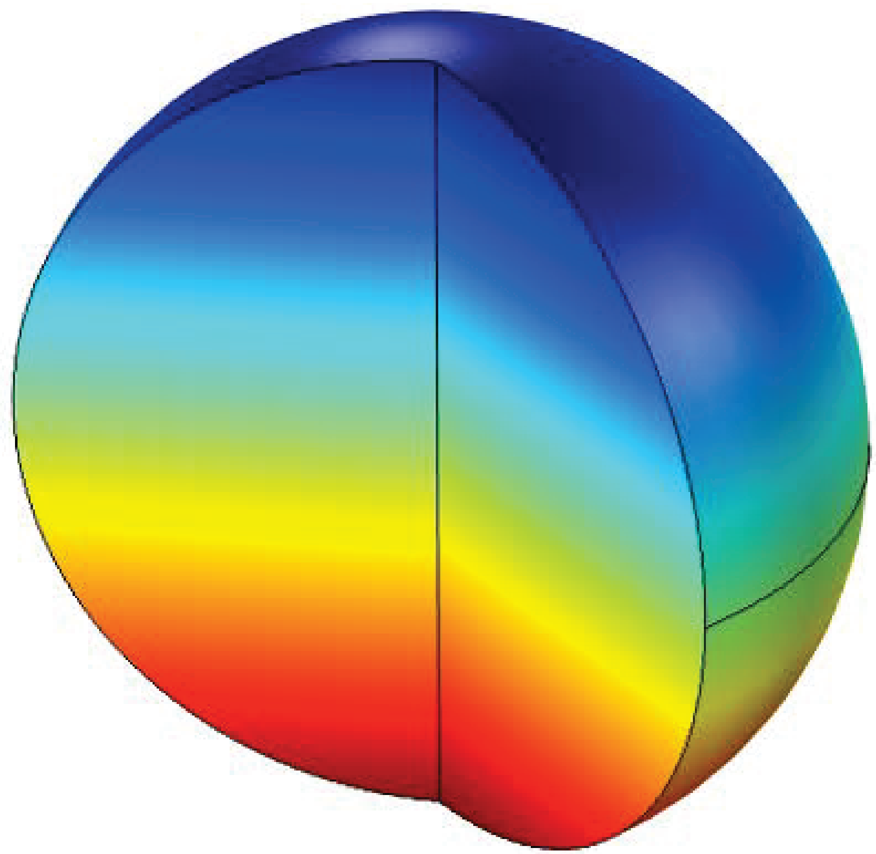

Fig. 3 Velocity potential, Φ, in the Inverted Far Exterior by COMSOL. The linear dependence upon z accords with the analytic solution.

Conclusions

C1. Kelvin's Inversion Theorem enables replacement of one flow in an unbounded domain by two flows, each of which is in a bounded domain;

C2. COMSOL enables simultaneous solution in the two bounded domains.

References

1. Kelvin, Lord Extrait d'une lettre de M. William Thomson à M. Liouville. *Journal de Mathématique Pures et Appliquées*, **10** (1845).
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