# Solving Time-Dependent Optimal Control Problems in Comsol Multiphysics

**Ira Neitzel, Uwe Prüfert, and Thomas Slawig**

### Problem Setting

Optimal control problems subject to time-dependent partial differential equations are challenging from the viewpoint of mathematical theory and even more so from numerical realization.

Essentially, there are two different approaches to solve such problems.

- "Discretize then Optimize": Transformation of the optimal control problem into a nonlinear programming problem by discretization.
- "Optimize then Discretize": Developing optimality conditions in function spaces that are discretized and solved.

- For certain classes of problems it is possible to derive optimality conditions in PDE form.
- The latter strategy then involves solving systems of PDEs.
- It hence suggests itself to apply specialized PDE software to solve these systems.
- We aim at applying COMSOL Multiphysics for optimization, taking advantage of the built-in routines to define, discretize and solve stationary and time-dependent PDEs via the finite element method.
- Time-dependent PDE control problems admit the typical feature of reverse time directions in the PDEs of the optimality systems.
- This additional difficulty needs to be taken into account when solving these problems numerically.

We consider the optimal control problem (P):

$$J(y,u)=\frac{1}{2}\int_Q (y-y_d)^2+\kappa u^2\,dxdt, \qquad (1)$$

subject to the parabolic PDE with distributed control

$$\left.\begin{aligned} y_t-\Delta y &= u && \text{in } Q\\ \partial_n y+\alpha y &= g && \text{on } \Sigma\\ y(t=0) &= y_0 && \text{in } \Omega \end{aligned}\right\}, \qquad (2)$$

**Consideration of boundary control problems also possible.**

### Theoretical Preparations

**Assumption 1.** In this setting, $\Omega\subset\mathbb{R}^N$, $N=1,2$, is a spatial domain with sufficiently smooth boundary $\partial\Omega$, $(0,T)$ is a non-empty time interval, $\Sigma:=\partial\Omega\times(0,T)$, and $Q:=\Omega\times(0,T)$. Moreover, we consider functions $g\in L^2(\Sigma)$ and $y_0\in L^2(\Omega)$ and controls $u\in L^2(Q)$.

A short formulation of the model problem with control $u$ and state $y$ then reads

$$\min J(y,u) \text{ subject to } (2) \qquad (P)$$

**Theorem** (Solvability of the state equation) For any triple $(f,g,y_0)\in L^2(Q)\times L^2(\Sigma)\times L^2(\Omega)$ the initial-boundary value problem

$$\begin{aligned} y_t-\Delta y &= f \text{ in } Q,\\ \partial_n y+\alpha y &= g \text{ on } \Sigma,\\ y(t=0) &= y_0 \text{ in } \Omega \end{aligned}$$

admits a unique solution

$$y\in W(0,T):=\{y\in L^2(0,T;H^1(\Omega))|y_t\in L^2(0,T,H^1(\Omega)^*)\}.$$

**Theorem** (Existence of an optimal solution) Under Assumption 1 and for $J$ defined in (1), and arbitrary $\kappa>0$, the optimal control problem defined in $(P)$ admits a unique optimal control $u^*\in U=L^2(Q)$.

**Theorem** (Optimality system) Let $u^*\in U=L^2(Q)$ be the optimal control of Problem $(P)$ and let $y^*$ denote the associated optimal state. Then there exists an *adjoint state* $p\in W(0,T)$ as weak solution of

$$\left.\begin{aligned} -p_t-\Delta p &= y^*-y_d && \text{in } Q\\ \partial_n p+\alpha p &= 0 && \text{on } \Sigma\\ p(t=T) &= 0 && \text{in } \Omega \end{aligned}\right\}, \qquad (3)$$

and the gradient equation

$$\kappa(u^*-u_d)+p=0 \qquad (4)$$

is fulfilled for almost all $(x,t)\in Q$.

More details: [3], [1]

### Strategies to deal with the reverse time directions

- Somewhat classical approach: sequentially solving the state and adjoint equation, updating the control in a gradient based optimization algorithm, cf. [2] for an implementation in COMSOL Multiphysics
- Alternative: Treating the coupled optimality system in the whole space-time cylinder by interpreting the time variable as an additional space variable.

### Treating the Reverse Time directions by Simultaneous Space-Time Discretization

- Insert gradient equation $(4)$ into state equation
- Interpret $Q$ as spatial domain of dimension $N+1$ with boundary $\Sigma\cup\Omega\times\{0\}\cup\Omega\times\{T\}$

$$\left.\begin{aligned} y_t-\Delta y &= u_d-\tfrac{1}{\kappa}p\\ -p_t-\Delta p &= y-y_d \end{aligned}\right\} \text{ in } Q, \qquad \left.\begin{aligned} \partial_n y+\alpha y &= g\\ \partial_n p+\alpha p &= 0 \end{aligned}\right\} \text{ on } \Sigma$$

$$\begin{aligned} y &= y_0 && \text{in } \Omega\times\{0\}\\ p &= 0 && \text{in } \Omega\times\{T\}. \end{aligned}$$

### An example in 2D

The space-time domain is defined by

$$Q=(0,\pi)^2\times(0,\pi)\subset\mathbb{R}^3$$

and the functions $y_d$, $u_d$, and $g$ are given by

$$y_d=\sin(x_1)\sin(x_2)\sin(t)-\cos(x_1)\cos(x_2)-2\cos(x_1)\cos(x_2)(\pi-t),$$

$$u_d=\sin(x_1)\sin(x_2)\cos(t)+2\sin(x_1)\sin(x_2)\sin(t)+\frac{1}{\kappa}\cos(x_1)\cos(x_2)(\pi-t),$$

$$g=-\vec{n}\sin(t)(\sin(x_1),\sin(x_2))^T,$$

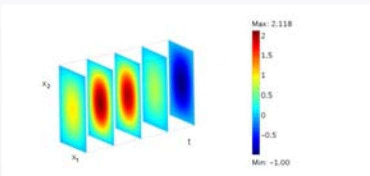
Moreover, $\alpha=0,\kappa=0.01$ are given.

### Optimal solution

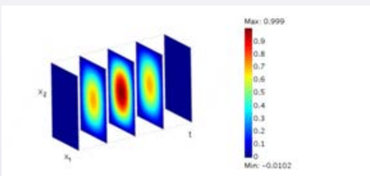
$$\begin{aligned} y^*(x_1,x_2,t) &= \sin(x_1)\sin(x_2)\sin(t)\\ u^*(x_1,x_2,t) &= \sin(x_1)\sin(x_2)(\cos(t)+2\sin(t))\\ p^*(x_1,x_2,t) &= \cos(x_1)\cos(x_2)(\pi-t), \end{aligned}$$

### Parts of a COMSOL Multiphysics Script:

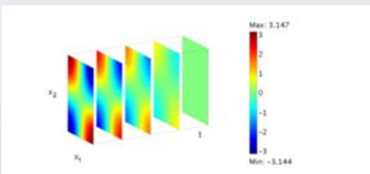
```
fem.equ.ga = { { {'-yx1' '-yx2' '0'};
               {'-px1' '-px2' '0'}} };
fem.equ.f  = { {'-ytime+u' 'ptime+y-yd(x1,x2,time)'} };
fem.bnd.r  = { {'y-y0' 0} {0 'p'} {0 0} {0 0} };
fem.bnd.g  = { {0 0} {0 0};
            {'g1(x1,time)-alpha*y' '-alpha*p'}
            {'g2(x2,time)-alpha*y' '-alpha*p'} };
```




$u_h$




$y_h$




$p_h$

| hauto | $\Vert u^*-u_h\Vert_Q$ | $\Vert y^*-y_h\Vert_Q$ |
|---|---|---|
| 7 | $3.1710\cdot10^{-1}$ | $4.7920\cdot10^{-3}$ |
| 6 | $1.7107\cdot10^{-1}$ | $2.3017\cdot10^{-3}$ |
| 5 | $5.0385\cdot10^{-2}$ | $5.4455\cdot10^{-4}$ |

Table 1: Errors to the 2D example, adaptive solver

### Conclusion

We have succesfully applied the finite element package COMSOL Multiphysics to simple time-dependent optimal control problems subject to PDE constraints by utilizing an Optimize then Discretize strategy.

- The introduced strategy works reasonably well for our simple example problems.
- We take advantage of the fact that optimality conditions can be formulated as a PDE.
- The method we use is easily implementable and may well serve as a first step towards optimizing a given goal without the use of specialized optimization routines.
- The approach does not substitute the use of specialized optimization software.
- Elliptic solvers are used for time-dependent parabolic control problems, which may cause instability problems.

## References

[1] J. L. Lions. *Optimal Control of Systems Governed by Partial Differential Equations.* Springer-Verlag, Berlin, 1971.

[2] Ira Neitzel, Uwe Prüfert, and Thomas Slawig. Strategies for time-dependent PDE control using an integrated modeling and simulation environment. part one: problems without inequality constraints. Technical Report 408, Matheon, Berlin, 2007.

[3] J. Wloka. *Partielle Differentialgleichungen.* Teubner-Verlag, Leipzig, 1982.

Contact: Ira Neitzel[1,*], Uwe Prüfert[2,**], Thomas Slawig[2].

Institute: [1]Technische Universität Berlin and *DFG priority program SPP 1253 Optimization with differential equations, **DFG research center MATHEON;

[2]Christian-Albrechts-Universität zu Kiel, DFG Cluster of Excellence The Future Ocean, DFG priority progam SPP 1253 Optimization with differential equations

Email: neitzel@math.tu-berlin.de,
pruefert@math.tu-berlin.de
ts@informatik.uni-kiel.de