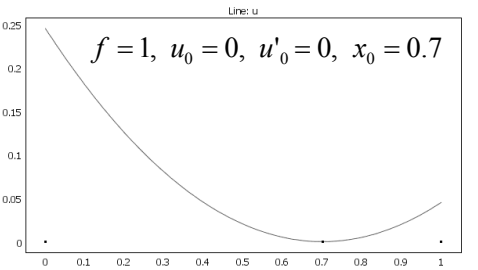
Solving a problem constrained only at an internal point

Problem definition

We want to solve the following problem

$$u'' = f \quad \text{in} \quad \Omega = (0,1)$$
$$u(x_0) = u_0$$
$$u'(x_0) = u_{x0} \quad \text{with} \quad x_0 \notin \partial\Omega$$



In this formulation the problem does not constitute a boundary value problem as it is conventionally solved with COMSOL because neither the the function u nor its derivative u‘ or a combaintion of both are specified at the boundary (no Dirichlet, Neumann or Cauchy-type boundary conditions).

Nevertheless, the problem can be solved easily with COMSOL, for instance in one of the following ways:

a) integration coupling of Neumann conditions, point constraint of u
b) as an initial value problem in the time domain

Way 1: Integrating of Neumann conditions

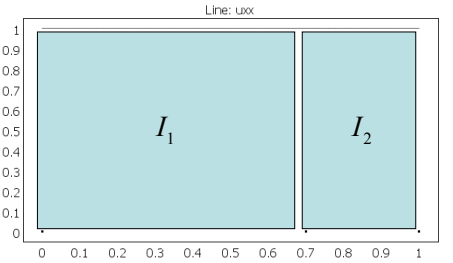
Since the second derivative is known in the whole domain and the first derivative in a point, it is straightforward to integrate the first derivative (Neumann conditions in both boundary points.

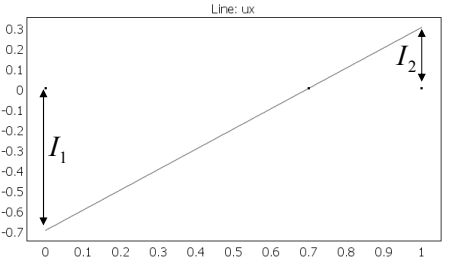
$$\left.\frac{\partial u}{\partial n}\right|_{x=0} = u'(x_0) - I_1$$

$$\left.\frac{\partial u}{\partial n}\right|_{x=1} = -\left(u'(x_0) + I_2\right)$$

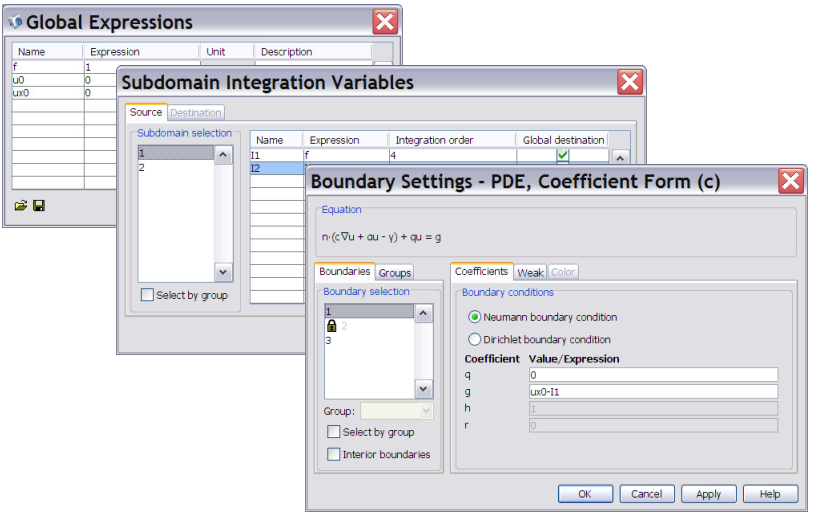
$$I_1 = \int_0^{x0} u''dx$$

$$I_2 = \int_{x0}^{1} u''dx$$





Implementation

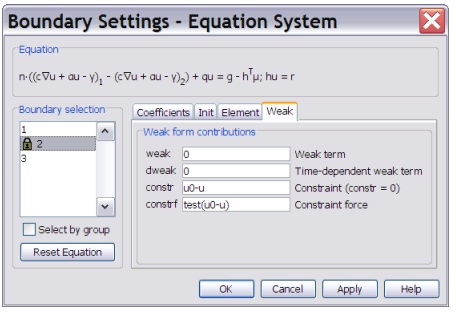


Constraining u in a point

With the Neumann conditions as just described our problem is not yet unique.

In order to make it, we need to constrain the unknown function in one point – the internal point x0. This is best done by using a weak term.

To find the proper syntax, you may for convenience check how a potential fixed in one point electrostatic mode translates into the equation settings → point → weak.



Way 2: Initial value problem

You can also look at our problem as an initial value problem and solve it as a global equation (ODE). You don‘t even need to define a space domain.

Note that t=0 corresponds to x0 and you need to „shoot“ into both directions to cover the given interval.