Numerical Study of Navier-Stokes Equations in Supersonic Flow over a Double Wedge Airfoil using Adaptive Grids

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Supersonic Flight

Shock wave formed on supersonic flight (Courtesy: Ensign John Gay, US Navy)
Supersonic Airfoils

**Supersonic Airfoil**
- Thinner cross-section
- Sharper leading and trailing edge

**Subsonic Airfoil**
- Thicker cross-section
- Rounded leading and trailing edge
Symmetrical Double Wedge Airfoil

\[ t_c = \frac{t_a}{c_a} : \{0.08, 0.1 \text{ and } 0.12\} \]

\( \alpha: \{0^0, 1^0, 2^0, 3^0, 4^0, 5^0, 8^0 \text{ and } 12^0\} \)

- \( c_a \): Chord Length
- \( t_a \): Thickness
- \( \alpha \): Angle of attack
• Section aerodynamic coefficients of an airfoil is defined below:

  • Coefficient of pressure \( C_p = \frac{p - p_\infty}{\frac{1}{2} \rho_\infty V_\infty^2} \)

  • Coefficient of Lift \( C_L = \frac{L}{\frac{1}{2} \rho_\infty V_\infty^2 c} \)

  • Coefficient of Drag \( C_D = \frac{D}{\frac{1}{2} \rho_\infty V_\infty^2 c} \)
Evaluation of Aerodynamic coefficients

- Defining geometrical and free stream parameters
  - Compressible Navier-Stokes theory
    - F.E.M meshing & solving in COMSOL
  - Shock-expansion theory
    - MATLAB code

Analysis of aerodynamic coefficients obtained
Compressible Navier-Stokes Theory

Non-conservative form:

Mass Conservation: \[ \frac{\partial \rho}{\partial t} + \rho \cdot \nabla \cdot (u) + (u \cdot \nabla) \rho = 0 \]

Momentum Conservation: \[ \rho \left( \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = \nabla \cdot \left[ -p \cdot I + \mu \cdot \left( (\nabla u) + (\nabla u)^T - \frac{2}{3} \cdot \nabla \cdot u \cdot I \right) \right] \]

Temperature equation: \[ \rho \cdot C_p \left( \frac{\partial T}{\partial t} + (u \cdot \nabla) T \right) = \nabla \cdot (k \cdot \nabla T) + \frac{T}{\rho} \cdot \left( \frac{\partial p}{\partial T} \right)_p \left( \frac{\partial p}{\partial t} + (u \cdot \nabla) p \right) + \nabla u : \left[ \mu \cdot \left( (\nabla u) + (\nabla u)^T - \frac{2}{3} \cdot \nabla \cdot u \cdot I \right) \right] \]

Ideal gas formulation: \[ p = \rho \cdot R \cdot T \]
**Numerical Simulation**

- **Boundary and Initial conditions:**

<table>
<thead>
<tr>
<th>Free stream parameters</th>
<th>Domain inlet values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mach numbers ($M_\infty$)</td>
<td>2.5</td>
</tr>
<tr>
<td>Temperature ($T_\infty$)</td>
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Domain outlet condition:

$$\nabla T \cdot \textbf{n} = 0$$
Grid generation
- Two level adaptive meshing feature on unstructured triangular mesh with first order element is implemented.
- Boundary-Layer cells are added to grid obtained from adaptive meshing.

Solver
- Viscous computation is initialised with prior solution obtained from Euler equations
- All the primary variables while are fully coupled and are solved using pseudo time stepping with a stationary solver.
- The convergence was determined by setting the relative tolerance to 0.01.

Adaptive mesh with boundary layer cells for Adaptive grid for $t_c = 0.1$ and $\alpha = 12^0$ case.
Shock-Expansion Theory

Oblique Shock

\[ \beta - \theta - M \text{ Relation} \]

\[ \tan(\beta) = \left( \frac{M_1^2 - 1 + 2 \cdot \lambda \cdot \cos \left( 4 \cdot \pi \cdot \delta + \frac{\cos^{-1}(\chi)}{3} \right)}{3 \cdot \left( 1 + \frac{(\gamma - 1)}{2} \cdot M_1^2 \right) \cdot \tan(\theta)} \right) \]

\[ \lambda = \left[ \left( M_1^2 - 1 \right)^2 - 3 \left( 1 + \frac{(\gamma - 1)}{2} \cdot M_1^2 \right) \left( 1 + \frac{(\gamma + 1)}{2} \cdot M_1^2 \right) \right] \cdot \tan^2(\theta) \]

\[ \chi = \left[ \left( M_1^2 - 1 \right)^3 - 9 \left( 1 + \frac{(\gamma - 1)}{2} \cdot M_1^2 \right) \left( 1 + \frac{(\gamma - 1)}{2} \cdot M_1^2 + \frac{(\gamma + 1)}{4} \cdot M_1^4 \right) \cdot \tan^2(\theta) \right]^{\frac{1}{2}} \]

Prandtl-Meyer Expansion

Isentropic expansion equation

\[ \theta = f(M_2) - f(M_1) \]

\[ f(M) = \sqrt{\frac{(\gamma + 1)}{(\gamma - 1)}} \cdot \tan^{-1} \left( \sqrt{\frac{(\gamma - 1)}{(\gamma + 1)} \cdot (M^2 - 1)} \right) - \tan^{-1} \left( \sqrt{M^2 - 1} \right) \]
Results

Coefficient of Lift

$C_L v/s \alpha$ (F.E.M Simulation)

$C_L v/s \alpha$ (SE-theory)
Coefficient of Drag

$C_D \nu/s \alpha$ (F.E.M Simulation) \hspace{2cm} C_D \nu/s \alpha$ (SE-theory)
Percentage Error estimation for \((t_c = 0.1)\)

\[
\frac{FEM - SE}{FEM} \times 100
\]

Percentage error of \(C_L\) \(v/s\) \(\alpha\)

Percentage error of \(C_D\) \(v/s\) \(\alpha\)
- **Results for specific case** \( t_c = 0.1 \):

  - Mach number plot at \( \alpha = 4 \)
  - Pressure plot at \( \alpha = 4 \)
  - Mach number plot at \( \alpha = 12 \)
  - Pressure plot at \( \alpha = 12 \)
Coefficient of pressure $C_p$
$(t_c = 0.1)$

$C_p \ (\text{SE-theory, } \alpha = 4)$

$C_p \ (\text{F.E.M, } \alpha = 4)$

$C_p \ (\text{SE-theory, } \alpha = 12)$

$C_p \ (\text{F.E.M, } \alpha = 12)$
CONCLUSION

• The solutions obtained from numerical simulation performed with FEM tool is in good agreement with shock-expansion theory.

• The difference in the values of coefficients obtained from SE-theory and compressible NS numerical simulation indicates the expected viscous and wake effects.

• The values of coefficients obtained from F.E.M simulation are only applicable for infinite span wing having airfoil section congruent to aerofoil designed in this current work.
THANK YOU & QUERIES ?